On Sets of Integers where Each Pair Sums to a Square

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Abstract

We discuss the problem of finding distinct integer sets $\{x_1, x_2, \ldots, x_n\}$ where each sum $x_i + x_j$, $i \neq j$ is a square, and $n \leq 7$. We confirm minimal results of Lagrange and Nicolas for $n = 5$ and for the related problem with triples. We provide new solution sets for $n = 6$ to add to the single known set. This provides new information for problem D15 in Guy's *Unsolved Problems in Number Theory*

1 Introduction

Let $\{x_1, x_2, \ldots, x_n\}$ be a set of n distinct non-zero integers. Erdos [1] and, independently, Moser, in [7], asked for examples of sets where $x_i + x_j$ is a square for all possible pairs of subscripts with $i \neq j$. This is considered in section D15 of Guy's well-known book [3].

For $n = 2$, there is only 1 pair and so the problem is easy. Pick any square p^2, then $\{x, p^2 - x\}$ is a suitable set.

For $n = 3$, we have a system of 3 equations in 3 unknowns,

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} p^2 \\ q^2 \\ r^2 \end{pmatrix} \tag{1}$$

This is easily solved to give

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} p^2 + q^2 - r^2 \\ p^2 - q^2 + r^2 \\ -p^2 + q^2 + r^2 \end{pmatrix} \tag{2}$$

Since squares are either congruent to 0 or 1 modulo 4, we cannot have more than one x_i of the form $4k + 1$ and more than one of the form $4k + 3$. Thus, at least one value must be even, either of the form $4m$ or $4m + 2$. If we had both types of odd numbers, then the sum of the even value plus the odds would give at least one value of the form $4i + 3$ which cannot be a square. Thus, the set can consist of at most one odd value, with the remainder even. Clearly, at most one value can be negative. This is true for all larger sets.

If we choose 3 even squares, we can get a set of even x_i, for example, $(2^2, 4^2, 8^2)$ gives the set $\{-22, 26, 38\}$, whilst, if we choose 2 odd squares and 1 even square we get a set with one odd element, for example, $(1^2, 2^2, 3^2)$ gives $\{-2, 3, 6\}$. Other choices lead to numbers with denominator 2, and if we multiply all x_i by 4 we preserve the paired squareness.

For $n = 4$, we follow the description given by Lagrange [4] and Nicolas [6]. We have 6 equations,

$$\begin{aligned} x_1 + x_2 &= p^2 & \qquad x_3 + x_4 &= t^2 \\ x_1 + x_3 &= q^2 & \qquad x_2 + x_4 &= u^2 \\ x_2 + x_3 &= r^2 & \qquad x_1 + x_4 &= v^2 \end{aligned} \tag{3}$$

and so $S = x_1 + x_2 + x_3 + x_4 = p^2 + t^2 = q^2 + u^2 = r^2 + v^2$.

Thus we, first, look for numbers S which have at least 3 different representations as the sum of 2 squares. Let the prime decomposition of S be

$$S = 2^i \, p_1^{n_1} p_2^{n_2} \dots p_k^{n_k} \; q_1^{m_1} q_2^{m_2} \dots q_l^{m_l}$$

where the p_i primes are $\equiv 3 \bmod 4$ and the q_i primes are $\equiv 1 \bmod 4$.

Then, if one of the n_i values is odd, the number of representations is 0. If all n_i are even, then the number of representations is

$$\frac{(m_1+1)(m_2+1)\ldots(m_l+1)+\delta}{2}$$

where $\delta = 1$ if all m_i are odd and $\delta = 0$ otherwise.

Having found a suitable S, we consider all possible groups of 3 different representations. The left hand set of equations in (3) is solved as in equation (2), and x_4 computed from the right-hand-sides. By changing the order of r and v we derive a second solution. All other permutations give one of these two basic solutions.

TABLE 1
Smallest sets of 4 elements

x_1	x_2	x_3	x_4
-40	65	104	296
-94	95	130	194
-88	88	137	488
-94	98	263	578
-190	239	290	386

TABLE 2
Smallest sets of 4 positive elements

x_1	x_2	x_3	x_4
2	359	482	3362
8	1016	1288	3473
162	567	1282	4194
2	167	674	6722
98	863	1346	5378

All this is very easily programmed using the freely available system Pari-GP, and can be run very quickly on any modern machine. These calculations lead

to the results in Tables 1 and 2, which give the 5 smallest general sets and strictly positive sets, respectively.

To measure the size of solutions, we use the l_1 norm $\sum |x_i|$ rather than $S = \sum x_i$, as used by Lagrange and Nicolas. The different measure of size means that the smallest set given by Lagrange and Nicolas is only the second smallest in Table 1.

2 n=5 sets

Suppose $\{x_1, x_2, x_3, x_4\}$ is a set such that all 6 pairs $x_i + x_j$, with $i \neq j$, are squares. We assume $x_1 < x_2 < x_3 < x_4$, and look for a fifth element x_5 to make a set of five elements.

We know p and q such that $x_1 + x_4 = p^2$ and $x_2 + x_4 = q^2$, and look for x_5 with $x_1 + x_5 = w^2$ and $x_2 + x_5 = y^2$. Thus $x_2 - x_1 = q^2 - p^2 = y^2 - w^2 = (y + w)(y - w)$. We loop over the divisors of $q^2 - p^2$, find suitable y and w, and then find $x_5 = w^2 - x_1$, checking if it satisfies the other conditions, namely $x_3 + x_5$ and $x_4 + x_5$ being square.

TABLE 3
Smallest sets of 5 elements

x_1	x_2	x_3	x_4	x_5
-4878	4978	6903	12978	31122
-2158	2258	4967	19058	33842
-5998	7847	9842	11474	30962
-878	882	7767	12114	48402
-1417	1586	5138	18578	45938

Tables 3 and 4 give the smallest sets (in the l_1 norm sense) of general integers and positive integers, respectively. Table 3 agrees with the results in Lagrange apart from the second and third sets being swapped, and the second smallest in Table 4 confirms Lagrange's speculation, since his table of results only contains one all-positive set, the first one in Table 4.

TABLE 4

Smallest sets of 5 positive elements

x_1	x_2	x_3	x_4	x_5
7442	28658	148583	177458	763442
32018	104882	188882	559343	956018
9122	104447	208034	348482	1295042
23458	82818	127863	228546	2149218
30818	56207	322018	910082	1946018

Now since the binomial coefficients $\binom{5}{2} = \binom{5}{3}$, there are the same number of subsets with 3 elements as with 2 elements, and we can relate them easily.

If we let $S = \sum x_i$, define the values $z_i = S/3 - x_i, i = 1, \ldots, 5$, then it is easy to see that the sum of any 3 non-repeated z_i values equals the sum of two of the x_i values, using the remaining subscripts eg. $z_1 + z_3 + z_4 = x_2 + x_5$. Thus $\{z_1, z_2, z_3, z_4, z_5\}$ forms a set of 5 elements where every triple sums to a square. If we have values of z_i which are rational we can scale by 9 to provide integer sets.

TABLE 5

Smallest sets with square triplets and 5 positive elements

x_1	x_2	x_3	x_4	x_5
92763	4914963	7559299	9945963	16308963
1039923	2292723	5649363	10128915	21847678
695883	2655435	18466923	40327563	62161518
33843	22986003	75168435	123438558	167502963
22906587	36372270	71091867	114486267	211586907

This problem was considered by Gill [2] in his wonderful book, and then in a more modern context by Wagon [8], who published a positive solution where each z_i has at least 20 digits. The computer search which produced Tables 3 and 4 can be easily adapted to search for positive sets - small sets with

negatives arise easily from small sets in Table 3. These triplet results are in Table 5.

These values are significantly smaller than that found by Wagon. In fact, the first set is given by Lagrange, right at the end of [5], though it is unclear which of Lagrange and Nicolas discovered the solution. The present results agree with the claim made that the first set is the smallest positive set giving square triples. This answers the question in Section D15 of Guy [3]. It is not obvious why this solution has lain unnoticed for so long. My 4 years of school French are enough to give a reasonable understanding of the papers.

Comparing Tables 4 and 5 we see that the values for positive sets with square triples are significantly higher than for sets with square pairs. A possible reason comes from the following analysis.

If we assume that $x_1 < x_2 < x_3 < x_4 < x_5$, then the smallest of the z_i values is z_5. So what we want to know is $\Pr(z_5 > 0)$. But this is equal to

$$\Pr(\, S > 3x_5 \,) = \Pr(\, x_1 + x_2 + x_3 + x_4 > 2x_5 \,)$$

$$= \Pr(\; 3x_1 + x_2 + x_3 + x_4 > 2(x_1 + x_5) \;)$$

$$= \Pr(\; (x_1 + x_2) \; + \; (x_1 + x_3) \; + \; (x_1 + x_4) > 2(x_1 + x_5) \;)$$

$$= \Pr(\, p^2 \; + \; q^2 \; + \; r^2 \; > \; 2s^2 \,)$$

for some positive integers p, q, r, s.

Now $p < q < r < s$, so what we want is $\Pr(\, y_1^2 \, + \, y_2^2 \, + \, y_3^2 \, > \, 2 \,)$ where $0 < y_1 < y_2 < y_3 < 1$. Unfortunately, the distribution of the y_i is unclear to me, so to proceed I assume they are uniformly distrubuted in $[0, 1]$.

Given a random point in the unit cube, there are 6 possible orderings of the coordinates, so the probability with $y_1 < y_2 < y_3$ is 1/6 of the probability for a general point.

The sphere $y_1^2 + y_2^2 + y_3^2 = 2$ cuts the unit cube at the obvious points $(1, 1, 0), (1, 0, 1), (0, 1, 1)$ and lies both inside and outside the cube, for example $(1, 1, 1)$ is outside. Thus the required probability is just the volume

within the cube but outside the sphere. The volume inside both cube and sphere is

$$\int_0^1 \int_0^1 \int_0^{\min\{1,\sqrt{2-x^2-y^2}\}} dz\,dy\,dx$$

which can be written as

$$\int_0^1 \int_0^{\sqrt{1-x^2}} dy\,dx \;+\; \int_0^1 \int_{\sqrt{1-x^2}}^1 \sqrt{2-x^2-y^2}\,dy\,dx$$

The first integral is easy and gives the answer $\pi/4$. The second is more difficult but possible, and was checked with a symbolic package giving $\pi(1-2\sqrt{2}/3)$. Thus the required probability (taking into account the 1/6 term) is $(\pi(8\sqrt{2}-15)+12)/72 \approx 1/172$. This gives a reasonable explanation of the rarity of positive triples.

3 n=6 Sets

We can apply the ideas of the last section to finding a value x_6 to give a set of 6 elements. There are 15 possible pairings. Lagrange calls a set satisfying only 14 a pseudo-solution. These are fairly easy to find and a large table is given in [5].

A special observation about such pseudo-solutions is that many have $S = \sum x_i$ being an integer square. Since S consists of three pairs which should each sum to a square, this leads to a consideration of representations of the form $p^2+q^2+r^2=s^2$. Lagrange uses the identity

$$(t^2+u^2+v^2+w^2)^2 = (t^2+u^2-v^2-w^2)^2 + 4(t\,w-uv)^2 + 4(tv+uw)^2$$

and a very inventive magic-square approach to find a representation which easily gives 13 out the the 15 pairs adding to a square. The remaining 2 pairs require two binary quartics to be made square. Each quartic is of the form

$$a\,G^4 + b\,G^3H + c\,G^2H^2 - b\,GH^3 + a\,H^4 = F^2,$$

though, obviously, with different (a,b,c) values.

Both quartics have many solutions and are thus birationally equivalent to different elliptic curves, which, on investigation, have large ranks of the order $3, 4, \ldots$. Thus, there will be lots of solutions with the possibility of a common solution, leading to a set satisfying all 15 identities. Lagrange completed the square, in the style of Euler, of one of the quartics, and used the resulting necessary condition to simplify the other quartic which he was again able to complete the square. This led to the solution set

$$\{-15863902, 17798783, 21126338, 49064546, 82221218, 447422978\}$$

In the current project, we searched the first quartic for values of (G, H) satisfying the first quartic and simply tested whether they satisfied the second. Some solutions give sets where values are repeated, which we discarded. The style of quartic allows us to only consider positive G, H values, and other coding tricks give a reasonably efficient code.

The program finds the solution of Lagrange very quickly and after some more computing, we find the following further solutions

$$\{-1126475417645550, 1274834760326775, 2236102802190450,$$

$$3990082124377234, 5054722548678450, 82588671432234450\},$$

$$\{-5098887366661368, 6849471293061768, 8279927229562632,$$

$$21501934179045768, 78740349884517393, 340192944008301832\}$$

and

$$\{-1177836637755448, 2476350655765448, 2723928921099848,$$

$$5744011161331073, 7858945782510152, 33438182171924552\}$$

It is perfectly possible that we have missed some smaller solutions. These results support Lagrange's conjecture that there are an infinite number of different solution sets for $n = 6$.

4 $n = 7$ Sets

Given the small number of solutions found for the $n = 6$ problem it would have been very surprising if we had found a solution for the $n = 7$ problem, which would require 21 pairs to be square.

Thus, we have concentrated in looking for sets which maximise the number of pairs. We first looked at the 4 solutions for the $n = 6$ problems and tried to extend them, using the method described before.

We found that, if we add 15945698 to the set found by Lagrange, we have 18 square pairs. None of the other 3 solutions give more than 17 square pairs in an extension.

Next, we used the Lagrange parameterisation which gives 14 square pairs and tried to add a seventh integer. This led to two 18 squares solutions

$$\{-256711392, 599109408, 741988233, 3222602992,$$

$$5845768992, 10931733792, 31619704608\}$$

and

$$\{-21145950, 21782754, 28598850, 56133175,$$

$$386338050, 873202050, 2468426850\}$$

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